Calculus via Symmetry

January 25, 2015

Abstract

In a sense, calculus is concerned with finding slopes of lines tangent to curves. While calculus traditionally focuses on manipulating algebraic expressions and taking limits, a lot can be gotten from symmetry and similar considerations, using a little algebraic manipulation but no limits.

Takeaway: We find equations of lines tangent to common curves. We also look at some integrals, major theorems, and miscellania.

There's more than one way to do these things. We'll show a few. Some of this material, marked as "fine print," is less intuitive but included for completeness.

Keywords: tangent lines, symmetry, derivatives, integrals

1 Definitions

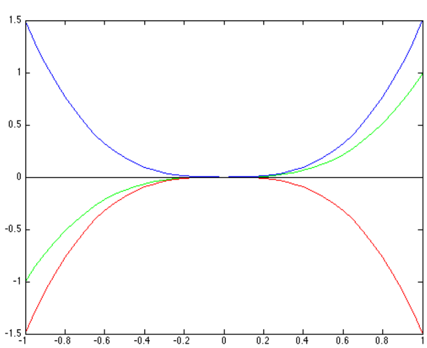
A **secant** line ℓ to a curve C touches C in two or more places. If C is the graph of a function and, for any two points P_1 and P_2 on C the segment ℓ joining P_1 and P_2 meet C only at P_1 and P_2 and is above C otherwise, then we say C is strictly **convex**. If the segment is strictly below C except at P_1 and P_2, then C is strictly concave.

A **tangent** line ℓ to a strictly convex curve C touches C exactly once and remains on one side of C. The definition still works where C is **locally convex**, *i.e.*, a connected part $\widehat{C}$ of C is convex and ℓ touches $\widehat{C}$ in the interior (not the endpoints of $\widehat{C}$).

1.1 Fine print

(Skip this at first.) We'll want tangent lines to be **unique**. This is critical in places.

Also, inflection points: If $f \leq g \leq h$ with $f(a) = g(a) = h(a)$, f strictly concave, h strictly convex, and f and h have common unique tangent ℓ at a, then ℓ is also the unique tangent line to g. For example, x^3 has an inflection point at zero but is sandwiched between $-1.5|x|^3$ and $+1.5|x|^3$.

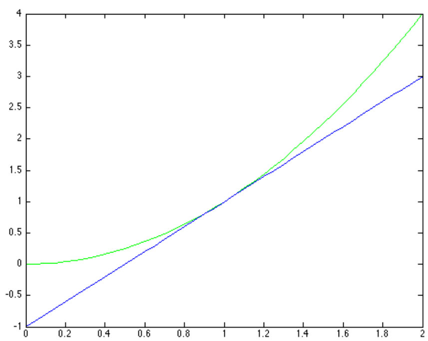


2 Tangents to Common Functions

2.1 Positive Integer Powers

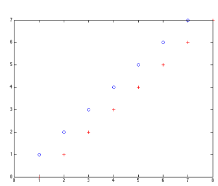
The square of a real y is zero only if $y = 0$ and positive otherwise. A straightforward generalization gives us tangent lines for all power functions.

- Show that $\ell(x) = 2x - 1$ is tangent to $f(x) = x^2$ at $x = 1$, by showing $\ell(1) = f(1)$ and $\ell(x) < f(x)$ for any $x \geq 0$. (Actually, for any x.)
- Consider the stretching symmetry $x = 3u$ and $y = 9v$. If $y = x^2$, what is v as a function of u? What happens to the point $(1, 1)$ of tangency, the y-intercept $(0, -1)$, and the tangent line $y = 2x - 1$ under this transformation?
- Stretch by general c instead of $c = 3$. Give the equation of a line tangent to $y = x^2$ at $x = c$.



- Repeat for $y = x^3$ and $y = x^4$ (considering only $x > 0$). See any patterns?
- Show that $x^n - nx + (n-1)$ is 0 if $x = 1$ and is positive for any other $x > 0$. Algebraically, this can be shown by induction or otherwise. We now consider a symmetric proof, using an example.
  - Show that $(x-1)^2(x^7 + 2x^6 + 3x^5 + \cdots + 7x + 8) = x^9 - 9x + 8$ by considering the sequence $(1, 2, 3, \ldots, 8)$ of coefficients. The graph is a triangular segment. The property we want is that if we shift the triangle to the right and subtract, we get a lot of cancelation:

1	2	3	4	5	6	7	8	
− 1	2	3	4	5	6	7	8	
1	1	1	1	1	1	1	1	−8



Repeat:

1	1	1	1	1	1	1	1	−8	
−	1	1	1	1	1	1	1	1	−8
1								−9	8

Note: The second stage is used to sum the geometric series $1 + x + x^2 + \cdots + x^{n-1}$, getting $\frac{1-x^n}{1-x}$. The two stages can be used to sum the series with increasing coefficients, getting, e.g.,

$$x^{n-2} + 2x^{n-1} + \cdots + n - 1 = \frac{x^n - nx + n - 1}{(1-x)^2}.$$

Substituting $x = 1/y$ and multiplying through by powers of y, we get

$$1 + 2y + 3y^2 + \cdots (n-1)y^{n-2} = \frac{1 - ny^{n-1} + (n-1)y^n}{(y-1)^2}.$$

If $|y| < 1$ and $n \to \infty$, we get $\frac{1}{(1-y)^2}$.

- So we have $nx - n + 1$ tangent to $y = x^n$ at $x = 1$. Apply the stretching transformation $x = 3u$ and $y = 3^n v$ and see what happens to the curve, point of tangency, and tangent line. State the general theorem.

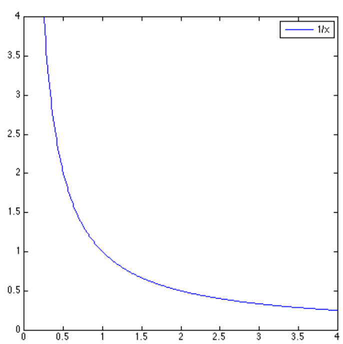
Some finer print:

- Show that $cx = x^n$ has zero as a root and some positive root if $c \neq 0$. Conclude that the tangent line at 0 is unique.
- Finally, show that x^2 is convex. That is, for $t \in (0,1)$ and any x_1, x_2, show $(tx_1 + (1-t)x_2)^2 \leq tx_1^2 + (1-t)x_2^2$. Use a symmetry to show that we can take $x_1 = 0$ and $x_2 = 1$, and switch t and $1-t$, getting $((1-t)0 + t \cdot 1)^2 < (1-t)0^2 + t \cdot 1^2$, or $t^2 < t$, which is obvious for $t \in (0,1)$.
- State the above for $x < 0$, even and odd powers separately, and appeal to symmetry rather than modify the above.
- Finally, for odd n, sandwich x^n between $\pm 2|x|^n$ to show that $y = 0$ is tangent to x^n at 0.

To get all polynomials, we need linearity of the derivative and of tangent lines. If we take linearity as a kind of symmetry, this is straightforward. See below for derivations of linearity in the tangent and integral domains.

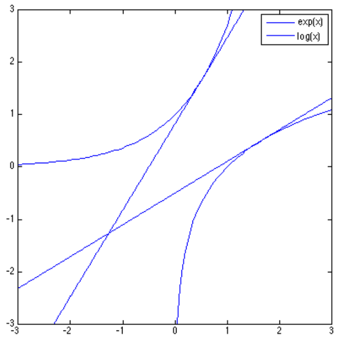
2.2 Reciprocals

The above dilation argument works on x^n for any real n, but the fact that $x^n \geq nx - (n-1)$ works for any n requires more tricks and gets progressively harder through n equal to a negative integer, reciprocal of an integer, or rational number. The general case can be handled with the chain rule, product rule, and logarithm rule, but we point out symmetries in some special cases, next.



- Above is the graph of $y = 1/x$, *i.e.*, $xy = 1$. Find a tangent line at some point P of your strategic choice, using a reflective symmetry.
- Try to find the tangent line at $(2, 1/2)$. Find an appropriate stretching symmetry that takes the curve to itself, preserves all tangencies, and moves the point of tangency from P to $(2, 1/2)$.
- Reflect about the line $y = x$ to handle the case $x^{1/n} - x/n + 1/n - 1$ for n a positive integer.
- For negative integers, using algebra, show that $x^{-2} + 2x - 3 \geq 0$ by transforming to $y^2 + 2/y - 3$ and then $y^3 + 2 - 3y$, and generalize to negative integer n.

2.3 Exponentials and Logs



The exponential function $\exp(x) = e^x$ is defined to satisfy

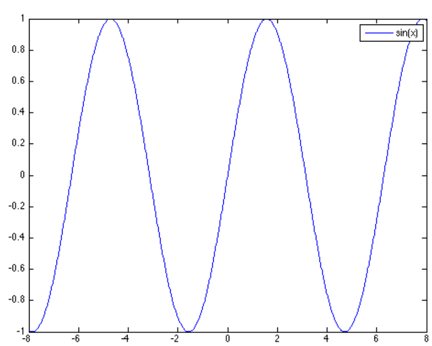
$$\begin{cases} e^{x+y} = e^x \cdot e^y \\ \text{tangent at } x = 0 \text{ is } 1 + x. \end{cases}$$

(So $e^0 = 1$, as expected for a zero exponent.) The inverse function is called the natural log, written $\ln(x)$ or $\log(x)$.

- Find a stretching/translation symmetry that preserves tangencies, takes the exponential function to itself, and moves the tangency point $(0, 1)$ to $(1/2, e^{1/2})$. What is the equation of the line tangent to e^x at $(1/2, e^{1/2})$? Leave in the form $m(x - a) + b$.
- Find a reflective symmetry that takes exp to log. What is the tangent to $\log(x)$ at $(e^{1/2}, 1/2)$? Leave in the form $m(x - a) + b$.

Conclude that the slope of the tangent to e^x at $x = a$ is e^a and the slope of the tangent to $\log x$ at $x = b$ is $1/b$.

2.4 Sines and Cosines



Consider a unit spiral in three space. A particle goes around the origin-centered, unit radius circle (the "unit circle") in the x-y plane at constant speed (parametrized by time, t), while advancing in the z direction proportional to t. At time $t = 0$, the position is $(1, 0, 0)$ and the point is moving in the y- and z- directions with equal speed; that is, the tangent line is $(1, 0, 0) + (0, 1, 1)s$ in parameter s. Then sine and cosine are defined as the y and x components of the particle's position, so the particle at time t is at $(\cos t, \sin t, t)$.

- Note that the line tangent to the unit circle at P is perpendicular to the radius at P. Find the equation of the line tangent to the spiral at point $(\cos t, \sin t, t)$.

- Project the sprial onto the y-z plane, getting $(\sin t, t)$ or, with y vertical and z horizontal, the familiar wavy sine curve $y = \sin z$. What is the tangent to the sine curve at $(\sin t, t)$? Repeat for $x = \cos z$ by projecting onto the x-z plane **and** by rotating and translating the spiral.

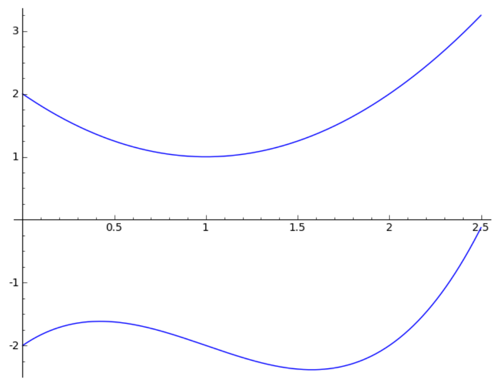
Conclude that the slope of the tangent to $\sin(x)$ at $x = a$ is $\cos(a)$ and the slope of the tangent to $\cos(x)$ at $x = b$ is $-\sin(b)$.

3 Integrals

Area under a curve should obey certain familiar properties. It should behave correctly under dilation, rotation, reflection, scaling and finite rearrangement. A 1×1 square has area 1 (whence rectangles and triangles have determined area from symmetries).

The finite rearrangement property leads to additivity of the integral, without saying (much) more. Our "area under a curve" will be signed properly. For example, if $f \le 0 \le g$, then the area between f and g is the signed area between 0 and g plus the area between f and 0, and the latter is the area between 0 and the positive function $-f$.

We do not attempt to be thorough in analyzing which curves have integrals. This is a deep subject beyond the scope of this activity.



3.1 Simpson and Newton-Cotes

Simpson's rule says that the area under any quadratic $q(x)$ over $[-1, 1]$ is $c_{-1}q(-1) + c_0q(0) + c_1q(1)$. We need to find the c's and argue that a fixed set of c's works for all q.

Background:

- The following are all linear: integration (area from graph of function), interpolation (graph of entire function from sufficiently many samples), and Simpson's formula. So a choice of c_{-1}, c_0, c_1 that works on $1, x, x^2$ works for any quadratic.

- Any three different points determines a quadratic and Simpson's formula uses three different points. So some choice of c_{-1}, c_0, c_1 works on any quadratic.

Now, to find the constants:

- If $q(x) = 1$, the area is 2 and evaluations are all 1, so we get $c_{-1} + c_0 + c_1 = 2$. Repeat for $q(x) = x$ to get a second equation in the c's, and show that $c_{-1} = c_1$.

- We want a third equation from $q(x) = x^2$, but we don't (yet) know the area under x^2. First show $\int_0^1 x^2\,dx = \frac{1}{2}\int_{-1}^1 x^2\,dx$ is the (unknown) coefficient c_1.

- Now consider scaling. Dilate x^2 by 2 horizontally and by 4 vertically and conclude $\int_0^2 x^2\,dx = 8\int_0^1 x^2\,dx = 8c_1$.
- On the other hand, apply Simpson to the translated function, getting

$$\int_0^2 x^2\,dx = \int_{-1}^1 (t+1)^2\,dt = 0\cdot c_{-1} + 1\cdot c_0 + 4\cdot c_1.$$

- So

$$\begin{cases} c_{-1} & + & c_0 & + & c_1 & = 2 \\ -c_{-1} & & & + & c_1 & = 0 \\ & & c_0 & - & 4c_1 & = 0 \end{cases}$$

  Conclude $c_{-1} = c_1 = 1/3$ and $c_0 = 4/3$.

Now, generalize.

- Show that Simpson gives the exact answer on x^3, too, even though it takes four values to determine a cubic.
- Repeat for x^4 over $[-2,2]$ with evaluation points $\{-2,-1,0,1,2\}$. That is, $\int_{-2}^2 q(x) = c'_{-2}q(-2) + c'_{-1}q(-1) + c'_0 q(0) + c'_1 q(1) + c'_2 q(2)$ for any quartic q. Use even symmetry to show $c'_{-2} = c'_2$ and $c'_{-1} = c'_1$. Use the known integral of 1 and the (now) known integral of x^2 to get two more equations. (Why are x and x^3 useless?) Finally, use dilation symmetry of x^4 to get the fifth and final equation and solve for the c''s.
- Repeat for higher powers.

3.2 Exponentials

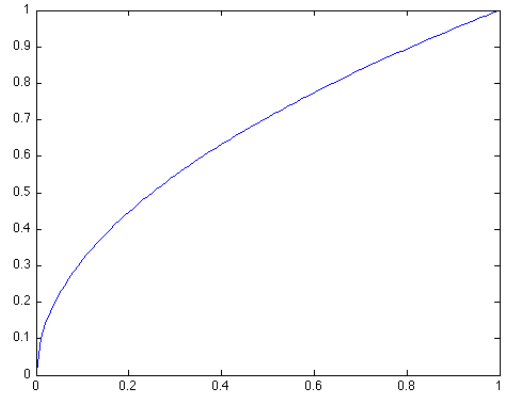
- Show $\int_{-\infty}^t e^x\,dx = e^t \int_{-\infty}^0 e^x\,dx$ by applying the symmetry of e^x consisting of translating and scaling.
- Conclude with $\int_a^b = \int_{-\infty}^b - \int_{-\infty}^a$.

Note: We can take $\int_{-\infty}^0 e^x\,dx = 1$ by definition of e.

A limit-centric approach would insist that the $-\infty$ in the integrand require limit analysis. But alternative analysis uses the self-similarity of e^x and is not much more than summing a geometric series like $.33333\ldots = 1/3$.

3.3 Rational Powers and Integration by Parts

We can get $\int_0^1 \sqrt{x}\,dx$ as $1 - \int_0^1 y^2\,dy$, by reflecting about $y = x$. That is, if $y = \sqrt{x}$, then $x = y^2$, and we integrate the latter. This is a restricted form of integration by parts: $\int y\,dx = xy - \int x\,dy$.



We can get $\int_1^t \frac{dx}{x} = \log t$ by using stretching symmetry. By definition of e and the logarithm, we have $\int_1^e \frac{dx}{x} = 1$. Then, by dilating x by c and contracting y by c, we preserve the curve and take a region under the curve to a similar region. That is,

$$\int_1^t \frac{dx}{x} = \int_c^{ct} \frac{dx}{x}.$$

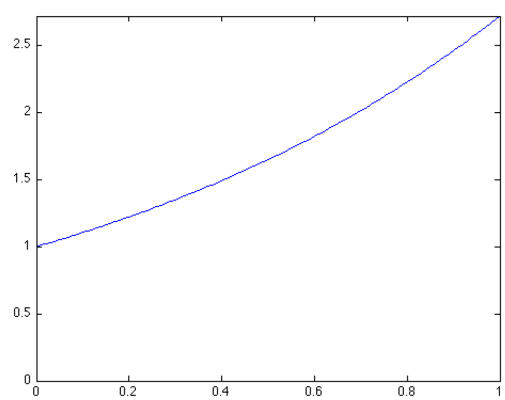
So, for any integer n, we have

$$\begin{aligned} 1 &= \int_1^e \\ &= \int_1^{e^{1/n}} + \int_{e^{1/n}}^{e^{2/n}} + \cdots + \int_{e^{1-1/n}}^{e} \\ &= n \int_1^{e^{1/n}}, \end{aligned}$$

whence $\int_1^{e^{1/n}} = 1/n$. Similarly, $\int_1^{e^{m/n}} = m/n$.

So the result follows for right endpoints of the form $e^{\mathbb{Q}}$, by definition of the log.

We can also hand $\int \log(x)\, dx$ by integration by parts. For example, $\int_0^1 e^x\, dx = e - 1$, by the above. This is the area between the curve $y = e^x$ and the x axis. The entire box below has area e, so the area between the curve $x = \log(y)$ and the y-axis is $e - (e - 1) = 1$.



3.4 Trigonometry

Ideally, we'd want to exploit spiral symmetry to handle cosine and sine, similar to what we did for exponentials. That is, instead of integrating sin or cos alone, integrate $\cos t + i \sin t = e^{it}$, using the spiral (i.e., shift and rotate) symmetry, $e^{i(t+s)} = e^{is}e^{it}$. Then we'd want $\int_{-\infty}^{b} e^{it}\,dt$ to be $e^{ib}\int_{-\infty}^{0} e^{it}\,dt$. Unfortunately, $\int_{-\infty}^{0} e^{it}\,dt$ is not defined, so this approach won't work (directly).

Mollification Instead of $\int_{-\infty}^{0} e^{it}\,dt$, compute $\int_{-\infty}^{0} e^{(i+\delta)t}\,dt$. For any $\delta > 0$, the integral exists and the integrand is self-similar. So we can compute any $\int_{a}^{b} e^{(u+iv)x} \cos t\,dt$ for non-zero u (and even infinite a and b where the integral makes sense) even if we can't compute $\int \cos t\,dt$.

If we are ultimately interested in $\int_{a}^{b} e^{it}\,dt$, we can choose $\delta = \epsilon/(b-a)$ so that $e\delta t = e^{a}(1 \pm \epsilon)$ on the range in question. Clearly this is a standard calculus question, involving limits, that we want to avoid.

Special intervals of integration Alternatively, we can content ourselves with $\int_{a}^{b} \cos x\,dx$ for $b-a$ of the form m/n for integers m and n.. First, we need to show that multiplication of complex numbers amounts to multiplying magnitudes and adding angles. Again, we content ourselves with rational fractions of a full rotation.

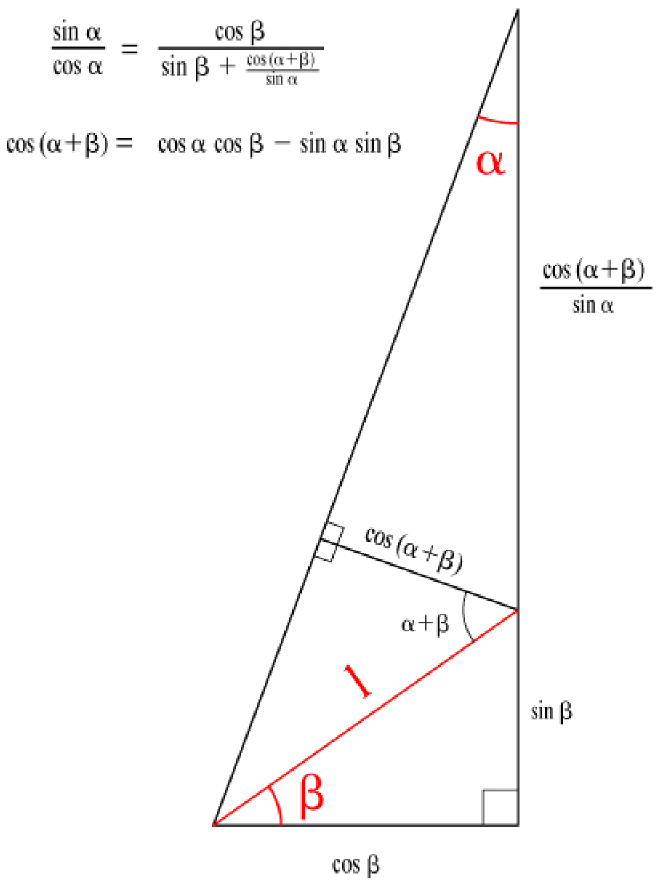
We need the fact that multiplying complex numbers adds their angles and multiplies their magnitudes. It's easy to multiply $(a+bi)(c+di) = ac - bd + (bc+ad)i$.

$$\|a+bi\|^2 \cdot \|c+di\|^2 = (a^2+b^2)(c^2+d^2) = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$

while

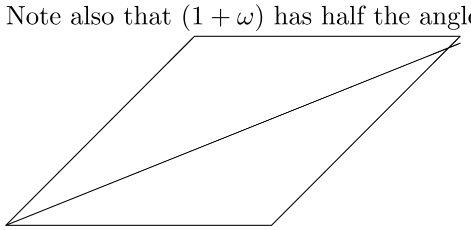
$$\|(a+bi)(c+di)\|^2 = \|ac-bd+(bc+ad)i\|^2 = (ac-bd)^2+(bc+ad)^2 = [a^2c^2+b^2d^2-2abcd]+[b^2c^2+a^2d^2+2abcd].$$

As for the addition of angles, by the above magnitude calculation and radial stretching symmetry, it is enough to consider complex units. A geometric angle sum formula is easy enough:



(See `http://math.uaa.alaska.edu/~smiley/trigproofs.html`.)

Note also that $(1+\omega)$ has half the angle as ω.



Also, if ω^2 is known to have twice the angle as ω, then $(1+\omega)^2 = 1 + 2\omega + \omega^2$ has the same angle as ω. Thus, if the double-angle formula works on ω, it also works on $1+\omega$. Since it works on i, it works on angles of the form $\pi 2^{-n}$ by induction.

Since angles add when complex numbers multiply and since $e^0 = 1 = \cos 0 + i \sin 0$, we can justify $e = \cos t + i \sin t$, where the choice of base is part of the definition of e.

Now, the integral. By definition of $\sin, \cos, \pi$, and radians (*i.e.*, the one free dilating constant), we have $\int_0^{\pi/2} \sin t\, dt = 1$. From periodicity of $(\cos t + i \sin t)$, the integral over an integer number of periods of $(\cos t + i \sin t)$ is 0 and we can get $\int_0^x$ for $x > \pi/2$ from $\int_0^{x-\pi/2}$. So we only need to consider $\int_0^x (\cos t + i \sin t)\, dt$ for $0 < x < \pi/2$. We will compute $\int_0^x (\cos t + i \sin t)\, dt$ for $0 < x < \pi/2$ of the form $x = \pi m/n$.

Put $\omega = e^{i\pi/n} = \cos(\pi/n) + i \sin(\pi/n)$. First, $I = \int_0^{\pi/n} (\cos t + i \sin t)\, dt$ comes from $I(1 + \omega + \omega^2 + \cdots +$

$\omega^{n-1}) = 2i$, since we can break $2i = \int_0^\pi (\cos t + i \sin t)\, dt$ into

$$\int_0^{\pi/n} + \int_{\pi/n}^{2\pi/n} + \cdots + \int_{\pi(1-1/n)}^{\pi} .$$

Summing the series, and noting that $\omega^n = -1$, we have $I = (1-\omega)i$. Then, similarly,

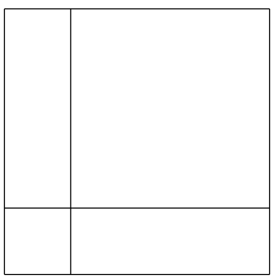
$$\int_0^x (\cos t + i \sin t)\, dt = \int_0^{\pi m/n} dt = I(1 + \omega + \omega^2 + \cdots + \omega^{m-1}) = I\frac{1-\omega^m}{1-\omega} = i(1-\omega^m) = i - i\cos x + \sin x.$$

4 Major Theorems

4.1 Products

Assuming we know how to handle x^2, compositions of functions, and linear combinations of functions, we can get products.

- Write fg in terms of squares, sums, differences, and scalings, using the identity $(x+y)^2 = x^2+y^2+2xy$. Derive $(fg)' = fg' + f'g$.



- Alternatively, we can look at dimensions, symmetry under exchanging f and g, symmetry under scaling, and the case of constant f. Suppose f has dimension volts-given-amps and g is meters-given-watts. (Or use students' names: Alice-given-Bob and Carol-given-Dave). What are the dimensions of $f', g', fg, (fg)', f^n, f^{(n)}$? Here $f^{(n)}$ means n derivatives of f.
- What are the possible ways to multiply together factors of the form $f^{(n)}$ and $g^{(n)}$ (using different or repeated n's) to get the correct dimensions of $(fg)'$?
- Suppose f is constant, so $(fg)' = fg'$. This means the coefficient of fg' must be 1 in the linear combination of terms of correct dimension. Now exchange f and g. What remains invariant? Conclude about the coefficients.
- What if $f = g$? Check for consistency or rederive the above results.
- Using the chain rule on log instead of the square, show $(\log(f))' = \frac{f'}{f}$. Apply also to g and fg to derive $(fg)'$.

4.2 Chain Rule and Inverse Functions

- Suppose f is strictly increasing. Reflect about the line $y = x$ to find the line tangent to f^{-1} at some point $f(a)$. (See plot in "Exponentials and Logs," above.)

For the general chain rule, it's hard to beat the intuition of Liebniz's $\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x}$ (which ultimately needs a limit). Still, here are some approaches based on convexity. All fine print below.

- For the general chain rule, suppose $y = f(x)$ and $z = g(y) = g(f(x)) = h(x)$, where f is convex and g is increasing and convex (locally). So h is convex. Consider the curve $(x, f(x), g(f(x)))$ in 3-space. Find a tangent line to $(x, f(x))$ in 2-space and extend to a plane $\pi_f = \{(x, y, z) : y = mx + b\}$ by taking all possible third coordinates. Similarly, find a tangent line to $(y, g(y))$ in 2-space and extend to a plane $\pi_g = \{(x, y, z) : z = m'y + b'\}$. Argue that $\pi_f \cap \pi_g$ is a line that, when projected to the x-z plane, is tangent to the curve $z = h(x)$ and has slope mm'. Uniqueness?

- Suppose $y = f(x)$ and $z = g(y) = g(f(x)) = h(x)$, where f is convex and g is increasing and convex (locally). So h is convex. If $\ell(x)$ is tangent to f at a, then $g(\ell(x)) \le g(f(x))$ with equality at a. Using above symmetries, find a line ℓ_h tangent to $g(\ell(x))$ in terms of tangents to $g(x)$ and argue that ℓ_h is tangent to h. (We need g convex only to see that h is convex so our program applies.)

- Using reflection symmetries, replace "convex" with "concave" and/or "increasing" with "decreasing."

- Show that if g has a strict minimum at $f(a)$ (so g is not monotone, and the constant $g(f(a)) = h(a)$ is tangent to g at $f(a)$), then the constant $h(a)$ is tangent to h at $x = a$.

- This leaves the common case of, say, $g(y) = y^3$ or $y^3 \pm y$ at $f(a) = 0$. Apply our definitions for inflection points. So, if $g(y) = y^3 \pm y$, our definition immediately transforms to y^3 with horizontal tangent for g. Assuming f is increasing, 0 lies between $g(f(x))$ and $g(-f(x))$. (If f has a minimum at $f(a) = 0$, then $g(\ell(x)) = g(f(a)) \le g(f(x))$.)

- Work out uniqueness.

4.3 Convexity Properties

We'll need some of these convexity properties.

- If f and g are convex (not necessarily strictly), then so is the sum.

- If *one* of f and g is strictly convex, then so is the sum.

- Suppose $y = f(x)$ and $z = g(y) = g(f(x)) = h(x)$. Suppose g is increasing and convex and that f is convex. Show that h is convex.

4.4 Linearity

The goal is to show that, if ℓ_f is the line tangent to f at a point a, ℓ_g is tangent to g at a, and c is a constant, then $\ell_f + c\ell_g$ is tangent to $f + cg$.

Linearity is much easier to see in the integral domain. It can be ported to the domain of tangent lines via the Fundamental Theorem of Calculus. Nevertheless, we give a direct approach here.

- Handle scaling by c via a simple vertical stretch.

- Suppose f and g are both locally convex. Show $\ell_f + \ell_g$ is tangent to $f + g$. (See below for uniqueness, which is harder.)

The remainder of this section is fine print.

- To show uniqueness of the tangent to $f + g$, suppose 0 is a common tangent to $f, g, f + g$ and ℓ is a different tangent line to $f + g$. Then ℓ twice intersects f and g, since their tangents are unique. It follows that ℓ lies above both f and g and, hence, the average, $(f + g)/2$. So $(f + g)/2$ has a unique tangent and so does $f + g$, by scaling.

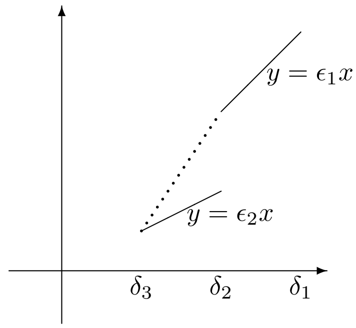


Figure 1: Solid segments are parts of lines of the form $y = \epsilon_i x$. We can arrange that $\epsilon_i = 1/i$, say, and that δ_i's decrease. The dotted segment shows how to form a convex polygon, g, larger than all the segments. If for all ϵ there exists δ with $x \in (0, \delta) \Rightarrow f(x) < \epsilon x$, then $f(x) \leq g(x)$ and g is convex with unique tangent line 0.

Even if f is locally convex and g is locally concave, $f + g$ may have infinitely many local oscillations. Our program, however, only considers functions that are locally strictly convex or locally concave. So assume f is strictly convex, g is strictly concave, and $f + g$ is strictly convex. Also assume that f and g have **unique** tangent lines.

- Show $g - \ell_g$ is strictly concave and $f - \ell_f$ is strictly convex with common tangent $y = 0$. Henceforth assume $\ell_f = \ell_g = 0$.
- Show $0 \leq f + g$ with equality only at a: If $0 = f(b) + g(b)$, then, by strict convexity of $f + g$, there's a c with $f(c) + g(c) < 0$. Again by convexity, $f + g$ lies on or below the secant line ℓ at 0 and c, which has non-zero slope. Suppose $0 < c$, so ℓ has negative slope. Then $g \leq f + g \leq \ell$ and $f \leq 0$, contradicting the assumption that g's tangent is unique.
- Show that 0 is a unique line tangent to $f + g$: If mx is also tangent, then $f(x) \geq f(x) + g(x) \geq mx$. This contradicts uniqueness of f's tangent line.

5 ...Versus Traditional Definition

Without loss of generality, consider functions f with $f'(0) = 0$.

- Show that if a function f is differentiable in our convexity-based sense, then it is also differentiable in the ϵ-δ sense. Hint: If f is convex with unique tangent 0 at 0, then $f(x)$ on $x > 0$ lies below each line ϵx secant at the points 0 and $(\delta, f(\delta))$.
- Conversely, show that if a function is differentiable in the ϵ-δ sense, then it is sandwiched between a convex and concave function with common tangent line 0 at 0. See Figure 1.